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A new source for a brane cosmological constant from a modified gravity model in the bulk

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Abstract

We show that a four-dimensional equation of state for a cosmological constant term arises from a perfect fluid in the bulk in the context of a gravity model where the scalar curvature is non-minimally coupled to the perfect fluid Lagrangian density. The four-dimensional theory is fully determined from the induced equations on the brane, subject to the boundary conditions derived across the brane.

I. INTRODUCTION

Braneworld scenarios are an interesting development in what concerns gravity models and their cosmological implications [1]. Most often these scenarios assume, given observational constraints as well as theoretical assumptions, that the bulk space is empty except for a cosmological constant. However, more recently the implications of having vector and scalar fields in the bulk were studied in connection with Lorentz symmetry [2] and gauge symmetry breaking [3].

In this paper, we consider the presence of a perfect fluid in the bulk in the context of a five-dimensional braneword model where the scalar curvature couples non-minimally to the Lagrangian density of the perfect fluid. In (3+1) dimensions this class of models with Lagrangian density of the form [4]

$$\mathcal{L} = \alpha f_1(R) + (1 + \lambda f_2(R))\, \mathcal{L}_M \ , \tag{1}$$

where $f_1(R)$ and $f_2(R)$ are generic functions of the scalar curvature, was shown to exhibit interesting features which allow one to address problems such as the rotation curves of galaxies without the need of dark matter (see Ref. [4] and references therein) and the Pioneer anomaly (see Ref. [5, 6] and references therein). The stability of these models has been examined in Ref. [7]. Other studies on their implications included their impact on stellar equilibrium and the analysis of their corresponding PPN parameters, which were studied in Refs. [8, 9], respectively.

Recently there has also been interest in the conformal equivalence between $f(R)$ theories and Einstein gravity non-minimally coupled to a scalar field in the context of braneworlds [10]. The expected increasingly higher order of the discontinuity of the geometric quantities across the brane with the increasing power in R of $f(R)$ is solved by enforcing continuity of the metric to correspondingly higher-order derivatives. Here, however, we will not impose further continuity conditions on the intervening fields, allowing for the discontinuity of the second derivative of the metric across the brane and orthogonal to its surface, despite also obtaining an increase in the power of R.

Crucial in the setting of our problem is a suitable implementation of the Israel matching conditions in the presence of bulk fields in order to extract the boundary conditions, both for gravity and the matter fields, which the induced equations of motion on the brane must satisfy. The method to be employed here was first introduced in Ref. [11] and further

developed in Refs. [2, 3]. For completeness and clarity, the more involved technical details of our method are presented in the Appendix. As we shall see, and rather remarkably, the projection of the bulk perfect fluid induces on the brane a new cosmological constant term. This new source for a brane cosmological constant opens quite interesting perspectives for inflation at the early universe and for acceleration at the late time expansion of the universe. This result suggests that a perfect fluid in the bulk space may have a bearing on the cosmological constant problem on the brane.

This paper is organized as follows. In section II we present our model and work out a suitable Lagrangian density for a perfect fluid. This development extends the approach of Hawking and Ellis [12] to the bulk space. In section III, we work out the matching conditions across the brane and derive the equations of motion therein induced. A derivation of the Gauss-Codacci relations is also presented in the Appendix for completion. Section IV contains our results and section V our conclusions.

II. A MODIFIED GRAVITY MODEL IN THE BULK

A. The Einstein Equations

We consider the particular case of the action discussed in Ref. [4]. We set $f_1(R) = f_2(R) = R$ and introduce a cosmological constant as follows

$$S = \int d^5x\sqrt{-g}\left[M^3_{P(5)}R + (1+\lambda R)\mathcal{L}_M - 2\mathcal{L}_\Lambda\right] \ . \tag{2}$$

Here $M_{P(5)}$ is the five-dimensional Planck mass, $\mathcal{L}_M$ and $\mathcal{L}_\Lambda$ are respectively the matter and the cosmological constant Lagrangian densities.

We define the stress-energy tensor as usual

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g^{\mu\nu}} \ . \tag{3}$$

For convenience, we define also the vacuum energy tensor as

$$\Lambda_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_\Lambda\right)}{\delta g^{\mu\nu}} \ , \tag{4}$$

assumed to be of the form

$$\Lambda_{\mu\nu} = \Lambda_{(5)}g_{\mu\nu} + \sigma\delta(N)(g_{\mu\nu} - N_\nu N_\mu) \ , \tag{5}$$

so as to include both the bulk vacuum value $\Lambda_{(5)}$ and the brane tension σ. Here N_μ are the components of the unit five-vector orthogonal to the brane $\vec{e}_N = N^\mu \vec{e}_\mu$. Thus, in Gaussian coordinates, the cosmological constant tensor is given by:

$$\mathbf{\Lambda} = \left(\begin{array}{c|c} {}^{(\mathbf{4})}\mathbf{g}\left(\Lambda_{(5)} + \sigma\right) & \\ \hline & \Lambda_{(5)} \end{array} \right) , \tag{6}$$

The five-dimensional Einstein equation is obtained by varying the action with respect to the metric, finding that

$$M^3_{P(5)} G_{\mu\nu} - \frac{1}{2}\left(1 + \lambda R\right) T_{\mu\nu} + \Lambda_{\mu\nu} - \lambda\left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box - \lambda R_{\mu\nu}\right) \mathcal{L}_M = 0 \ . \tag{7}$$

B. A Perfect Fluid in the Bulk

Since the Einstein equation in Eq. (7) contains terms in $\mathcal{L}_M$, we must construct a Lagrangian density associated with a perfect fluid so that by Eq. (3) it yields

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \ , \tag{8}$$

where ρ is the energy density, p the pressure and u_μ the unit five-velocity of the fluid (tangent to the flow lines and thus time-like, $u_\mu u^\mu = -1$). For this purpose, we follow closely the procedure described in Ref. [12].

Let the perfect fluid Lagrangian density $\mathcal{L}_M$ be given by

$$\mathcal{L}_M = -\tau(1 + \epsilon) \ , \tag{9}$$

where τ is an auxiliary variable and ϵ is the internal energy of the fluid as well as a function of τ. Assuming that the fluid current vector $j^\mu = \tau u^\mu$ is conserved, $\nabla_\mu j^\mu = 0$, then $\delta(\sqrt{-g} j^\mu) = 0$ when the metric is varied. Then, from

$$\tau^2 = -j^\mu j^\nu g_{\mu\nu} = \frac{1}{g}\left(\sqrt{-g} j^\mu\right)\left(\sqrt{-g} j^\nu\right) g_{\mu\nu} \ , \tag{10}$$

it follows that

$$\begin{aligned} 2\tau\delta\tau \ &= \ -\frac{\delta g}{g^2}\left(\sqrt{-g} j^\mu\right)\left(\sqrt{-g} j^\nu\right) g_{\mu\nu} \\ &+ \frac{1}{g}\left(\sqrt{-g} j^\mu\right)\left(\sqrt{-g} j^\nu\right) \delta g_{\mu\nu} \end{aligned}$$

$$= \left(j_\mu j_\nu - j^\beta j_\beta g_{\mu\nu}\right)\delta g^{\mu\nu} \ , \tag{11}$$

and consequently that the variation of τ with respect to the metric is given by

$$\delta\tau = \frac{1}{2}\left(\tau g_{\mu\nu} + \tau u_\mu u_\nu\right)\delta g^{\mu\nu} \ . \tag{12}$$

Using the definition of the stress-energy tensor, Eq. (3), we find that

$$\begin{aligned} T_{\mu\nu} &= \mathcal{L}_M g_{\mu\nu} - 2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} \\ &= \left[\tau(1+\epsilon) + \tau^2\frac{d\epsilon}{d\tau}\right]u_\mu u_\nu + \tau^2\frac{d\epsilon}{d\tau}g_{\mu\nu} \\ &= (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \ , \end{aligned} \tag{13}$$

where we made the following identifications

$$\rho = \tau(1+\epsilon) \ , \qquad p = \tau^2\frac{d\epsilon}{d\tau} \ . \tag{14}$$

We have thus obtained the stress-energy tensor for a perfect fluid from the Lagrangian density $\mathcal{L}_M = -\rho$ and from the continuity equation $\nabla_\mu(\tau u^\mu) = 0$. For an alternative formulation where the Lagrangian density is identified with the pressure see, for instance, Refs. [13, 14].

To obtain the equation of motion for the perfect fluid, one could compute the divergence of the Einstein equation, Eq. (7). Alternatively, we will proceed directly from the Lagrangian density. For this purpose, we consider the action of the Lie derivatives on the fluid flow lines. Let $\gamma : [a,b]\times\mathcal{N} \to \mathcal{D} \subset \mathcal{M}$ be a congruence of flow lines, one through each point of $\mathcal{M}$, where $[a,b]$ is the interval of the parameter ascribed to the flow lines, $\mathcal{N}$ is some four-dimensional manifold and $\mathcal{D}$ is a small region of the five-dimensional spacetime manifold $\mathcal{M}$. The tangent vector to the flow lines is given by $\mathbf{U} = (\partial/\partial t)_\gamma$, with $t \in [a,b]$, which once normalized we identify with the fluid velocity

$$u^\mu = \frac{U^\mu}{\sqrt{-g_{\alpha\beta}U^\alpha U^\beta}} = \frac{U^\mu}{|U|} \ . \tag{15}$$

The action S is required to be stationary when the flow lines are varied. Variations with respect to the flow lines, which we here represent by Δ, amount to variations along the corresponding tangent vectors. Considering $\gamma(r,[a,b],\mathbb{R}^4)$, where r is the parameter that selects different congruences of flow lines, then $\Delta\mathbf{U} = \mathcal{L}_\mathbf{V}\mathbf{U}$, where $\mathbf{V} = (\partial/\partial r)_\gamma$ and $\mathcal{L}_\mathbf{V}$ is the Lie derivative along $\mathbf{V}$. Since $\Delta(u^\mu|U|) = \mathcal{L}_\mathbf{V}U$, then

$$\Delta u^\mu = \frac{1}{|U|}\left(\mathcal{L}_\mathbf{V}U^\mu - u^\mu\Delta|U|\right) \ . \tag{16}$$

Moreover,

$$\Delta|U| = -\frac{1}{|U|}g_{\alpha\beta}U^{\alpha}\Delta U^{\beta} = -g_{\alpha\beta}u^{\alpha}\mathcal{L}_{\mathbf{V}}U^{\beta} \tag{17}$$

and

$$\mathcal{L}_{\mathbf{V}}U^{\beta} = V^{\sigma}U^{\mu}{}_{;\sigma} - U^{\sigma}V^{\mu}{}_{;\sigma} \ , \tag{18}$$

and consequently

$$\Delta u^{\mu} = V^{\sigma}u^{\mu}{}_{;\sigma} - u^{\sigma}V^{\mu}{}_{;\sigma} - u^{\mu}u^{\beta}V_{\beta;\sigma}u^{\sigma} \ . \tag{19}$$

From the conservation of the fluid current vector $j^{\mu}{}_{;\mu} = 0$, it follows that $\Delta(j^{\mu}{}_{;\mu}) = 0 = (\Delta j^{\mu})_{;\mu}$ which, using Eq. (19) and integrating along the flow lines, yields

$$\Delta\tau = (\tau V^{\beta})_{;\beta} + \tau V_{\beta;\alpha}u^{\beta}u^{\alpha} \ . \tag{20}$$

Thus, in the Lagrangian density, τ varies so that the associated current vector j^{μ} is conserved.

Finally, the condition for the stationarity of the metric yields

$$\begin{aligned}
\frac{\partial S}{\partial\tau} &= \int\sqrt{-g}d^5x\left\{\frac{d\mathcal{L}}{d\tau}\Delta\tau\right\} \\
&= \int\sqrt{-g}d^5x\left\{-(1+\lambda R)\left[1+\frac{d(\tau\epsilon)}{d\tau}\right]\Delta\tau\right\} \\
&= \int\sqrt{-g}d^5x\left\{-(1+\lambda R)\left[1+\frac{d(\tau\epsilon)}{d\tau}\right]\left[\nabla_{\mu}\left(\tau V^{\mu}\right)+\tau\left(\nabla_{\beta}V_{\mu}\right)u^{\mu}u^{\beta}\right]\right\} = 0 \ .
\end{aligned} \tag{21}$$

Integrating by parts and using the Stokes theorem to discard the surface terms, we find that

$$\begin{aligned}
\int\sqrt{-g}d^5x&\left\{\tau\nabla_{\mu}\left[(1+\lambda R)\left(1+\frac{d(\tau\epsilon)}{d\tau}\right)\right]\right. \\
&\left.+\nabla_{\beta}\left[(1+\lambda R)\left(1+\frac{d(\tau\epsilon)}{d\tau}\right)\tau u_{\mu}u^{\beta}\right]\right\}V^{\mu} = 0 \ ,
\end{aligned} \tag{22}$$

which holds for any vector $\mathbf{V}$. Therefore, the expression within curly brackets must vanish

$$\left\{\lambda\nabla_{\beta}R\left[1+\frac{d(\tau\epsilon)}{d\tau}\right]+(1+\lambda R)\left[\frac{d(\tau\epsilon)}{d\tau}\right]_{;\beta}\right\}\left(g^{\beta\mu}+u^{\beta}u^{\mu}\right)\tau \tag{23}$$

$$+\ (1+\lambda R)\left[1+\frac{d(\tau\epsilon)}{d\tau}\right]u^{\beta}(\nabla_{\beta}u^{\mu})\tau = 0 \ . \tag{24}$$

Using the equations in Eq. (14) and noticing that

$$\tau\left[\frac{d(\tau\epsilon)}{d\tau}\right]_{;\beta} = \left[\tau^2\frac{d\epsilon}{d\tau}\right]_{;\beta} = \nabla_{\beta}p \ , \tag{25}$$

we obtain the equation of motion for the perfect fluid in the bulk

$$[\lambda\,(\rho+p)\,\nabla_\beta R+(1+\lambda R)\nabla_\beta p]\,(g^{\beta\mu}+u^\beta u^\mu)+(1+\lambda R)(\rho+p)u^\beta\nabla_\beta u^\mu=0\;. \tag{26}$$

For $\rho+p\neq 0$, this equation can be rewritten as

$$u^\beta\nabla_\beta u^\mu=\frac{Du^\mu}{ds}=\frac{du^\mu}{ds}+\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta=f^\mu\;, \tag{27}$$

where f^μ can be regarded as an exterior force given by

$$f^\mu=-\left[\frac{\lambda}{1+\lambda R}\nabla_\beta R+\frac{1}{\rho+p}\nabla_\beta p\right]\left(g^{\beta\mu}+u^\beta u^\mu\right)\;. \tag{28}$$

Setting $\lambda=0$, one recovers the known equation of motion for a perfect fluid in General Relativity. Moreover, Eq. (28) agrees with the result obtained in Ref. [4] by taking the divergence of the Einstein's field equation. This indicates that the assumed conservation of the fluid current vector $j^\mu=\tau u^\mu$ is a consistent description of our physical system.

The continuity equation, which follows from the conservation of j^μ, provides the last equation and ensures that our problem is well defined. From Eq. (14) we find that

$$p=\tau^2\epsilon_{,\mu}\frac{1}{\tau_{,\mu}},\qquad \rho\frac{\epsilon_{,\mu}}{1+\epsilon}(p+\rho)=\rho_{,\mu}p\;. \tag{29}$$

The continuity equation

$$j^\mu{}_{;\mu}=\tau_{,\mu}u^\mu+\tau u^\mu{}_{;\mu} \tag{30}$$

is thus equivalent to

$$\rho_{,\mu}u^\mu+(p+\rho)u^\mu{}_{;\mu}=0\;. \tag{31}$$

For the ideal fluid, the gravitational field enters only in Eq. (26). Gravity does not appear in the velocity equation, Eq. (31), as the velocity is measured relative to freely moving observers [15].

III. THE INDUCED EQUATIONS ON THE BRANE

In this section, we derive the equations of motion induced on the brane. First, we rewrite the components of the equations of motion derived in the previous section in Gaussian normal coordinates. In our notation, the directions parallel to the brane are denoted by $A, B, \ldots$, while the normal direction is denoted by N so that the brane is localized at

$N = 0$. Using the results derived in Appendix A, we obtain the relevant components of the Einstein equation:

$$\begin{aligned} & M_{P(5)}^3\, {}^{(5)}G_{AB} - \frac{1}{2}\left(1 + \lambda\, {}^{(5)}R\right)\left[(\rho + p)u_A u_B + p g_{AB}\right] + \Lambda_{AB} \\ + & \ \lambda\left[\nabla_A \nabla_B + K_{AB}\nabla_N - g_{AB}\left(\Box + K\nabla_N + \nabla_N\nabla_N\right) - {}^{(5)}R_{AB}\right]\rho = 0 \ , \end{aligned} \tag{32}$$

$$\begin{aligned} & M_{P(5)}^3\, {}^{(5)}G_{AN} - \frac{1}{2}\left(1 + \lambda\, {}^{(5)}R\right)\left[(\rho + p)u_A u_N\right] \\ + & \ \lambda\left[\nabla_A \nabla_N - K_A^B \nabla_B - {}^{(5)}R_{AN}\right]\rho = 0 \ , \end{aligned} \tag{33}$$

$$\begin{aligned} & M_{P(5)}^3\, {}^{(5)}G_{NN} - \frac{1}{2}\left(1 + \lambda\, {}^{(5)}R\right)\left[(\rho + p)u_N^2 + p g_{NN}\right] + \Lambda_{(5)} \\ + & \ \lambda\left[\nabla_N \nabla_N \rho - g_{NN}\left(\Box + K\nabla_N + \nabla_N\nabla_N\right) - {}^{(5)}R_{NN}\right]\rho = 0 \ . \end{aligned} \tag{34}$$

Note that we will only keep the index indicating the corresponding dimension on the five-dimensional terms and whenever confusion is propitiated.

We then proceed to derive the matching conditions, which follow from the presence of the brane dividing into two regions the bulk spacetime and from the symmetries of the bulk fields across the two regions. Here we regard the brane as a $\mathbb{Z}_2$–symmetric surface of infinitesimal thickness 2δ and thus separating the bulk into two mirroring regions about $N = 0$. The symmetry about the brane establishes how bulk quantities relate on the two sides of the brane. Hence, vector components parallel to the brane are even in N, whereas normal components are odd. For tensor quantities this generalizes by considering each additional normal component to reverse the parity of the component with one less normal component. Accordingly, it follows that $u_A(N = -\delta) = u_A(N = +\delta)$, whereas $u_N(N = -\delta) = -u_N(N = +\delta)$. Likewise, $g_{AB}(N = -\delta) = g_{AB}(N = +\delta)$ and $K_{AB}(N = -\delta) = -K_{AB}(N = +\delta)$. Consequently, there will be quantities that are discontinuous across the brane and whose derivatives in N generate singular distributions at the position of the brane. Integration of these contributions in the coordinate normal to the brane allows to relate the induced geometry of the brane with the induced stress-energy therein localized. Hence, by extracting the singular contributions from the projected bulk equations and establishing the matching conditions, we obtain the equations of motion induced on the brane.

We equate the (AB) component of the Einstein equation using the Gauss-Codacci conditions in Eqs. (A20–A22) as well as Eq. (A13). Integrating along the N direction across

the position of the brane, we find that

$$
\begin{aligned}
-\sigma g_{AB} &= \lim_{\delta\to 0}\int_{-\delta}^{+\delta} dN\nabla_N\Big\{ M^3_{P(5)}\left(-K_{AB}+g_{AB}K\right) \\
&\quad +\lambda K\left[(\rho+p)u_Au_B+pg_{AB}\right]-\lambda\left(g_{AB}\nabla_N-K_{AB}\right)\rho\Big\},
\end{aligned} \tag{35}
$$

where upon integration by parts non-singular terms arise which vanish upon integration and which contribute to the effective equation of motion. For that we assume the energy density, the pressure and the fluid velocity to be continuous. This implies that only higher than second order derivatives in the N direction can be singular on the brane and consequently survive over the infinitesimal integration. From these considerations there follows the Israel matching condition

$$
\left(M^3_{P(5)}-\lambda\rho\right)\left(-K_{AB}+g_{AB}K\right)+\lambda K(\rho+p)\left(u_Au_B+g_{AB}\right)-\lambda g_{AB}\nabla_N\rho=-\frac{1}{2}\sigma g_{AB} \tag{36}
$$

which, upon taking the trace, yields

$$
\left(M^3_{P(5)}-\lambda\rho\right)K(-1+d)+\lambda K(\rho+p)\left(u_C^2+d\right)-\lambda d\nabla_N\rho=-\frac{d}{2}\sigma\ . \tag{37}
$$

Another useful result is

$$
\left(M^3_{P(5)}-\lambda\rho\right)K_{AB}=K\frac{1}{d}\left[g_{AB}\left(M^3_{P(5)}-\lambda\rho\right)+\lambda(\rho+p)\left(du_Au_B-g_{AB}u_C^2\right)\right]. \tag{38}
$$

Substituting Eq. (36) back in the (AB) component of the Einstein equation, we obtain

$$
\begin{aligned}
&\left(M^3_{P(5)}-\lambda\rho\right)\left[G_{AB}-KK_{AB}+\frac{1}{2}g_{AB}\left(K^2+K_{CD}K^{CD}\right)\right]+g_{AB}\Lambda_{(5)} \\
-&\left[\frac{1}{2}\left(1+\lambda\left\{R-K^2-K_{CD}K^{CD}\right\}\right)+\lambda K\nabla_N\right]\left[(\rho+p)\left(u_Au_B+g_{AB}\right)\right] \\
+&\left[\frac{1}{2}g_{AB}+\lambda\left(\nabla_A\nabla_B-g_{AB}\nabla_C^2\right)\right]\rho=0\ .
\end{aligned} \tag{39}
$$

From the (AN) component of the tensor equation we notice that

$$
G_{AN}=K_{AB|B}-K_{|A}=-\nabla_B\int dNG_{AB}=-\nabla_A\mathcal{T}_{AB}\ , \tag{40}
$$

where $\mathcal{T}_{AB}$ stands for the stress-energy tensor induced on the brane as given by the Israel matching condition in Eq. (36). If we impose conservation of energy on the brane, it follows that $G_{AN}=0$, which implies the condition

$$
\nabla_A\left[K(\rho+p)(u_Au_B+g_{AB})\right]-\left[g_{AB}\nabla_N-(K_{AB}-g_{AB}K)\right]\nabla_A\rho=0\ . \tag{41}
$$

Furthermore, equating the (NN) component of the Einstein equation and integrating along the N direction, we find that

$$0 = \lim_{\delta\to 0}\int_{-\delta}^{+\delta} dN\nabla_N\left\{\rho K + K\left[(\rho+p)u_N^2 + p\right]\right\} = -2K(\rho+p)u_C^2 \ . \tag{42}$$

Substituting back in the (NN) component of the Einstein equation, we find that

$$\begin{aligned}
&\left(M^3_{P(5)} - \lambda\rho\right)\frac{1}{2}\left(-R + K^2 - K_{CD}K^{CD}\right) + \Lambda_{(5)} \\
-&\left[\frac{1}{2}\left(1 + \lambda\left\{R - K^2 - K_{CD}K^{CD}\right\}\right) + \lambda K\nabla_N\right]\left[(\rho+p)\left(u_N^2 + 1\right)\right] \\
+&\left[\frac{1}{2} - \lambda\left(\nabla_C^2 + K\nabla_N\right)\right]\rho = 0 \ .
\end{aligned} \tag{43}$$

Moreover, substituting $\nabla_N\rho$ from the Israel matching condition for a time-like fluid velocity normalized so that $u_A^2 + u_N^2 = -1$, it follows that

$$\begin{aligned}
&\left(M^3_{P(5)} - \lambda\rho\right)\frac{1}{2}\left[-R + K^2\left(\frac{2}{d} - 1\right) - K_{CD}K^{CD}\right] + \Lambda_{(5)} \\
+&\left[\frac{1}{2}\left(1 + \lambda\left\{R - K^2 - K_{CD}K^{CD}\right\}\right) + \lambda K\nabla_N\right]\left[(\rho+p)u_C^2\right] \\
+&\left[\frac{1}{2} - \lambda\nabla_C^2\right]\rho - \frac{1}{d}\lambda K^2(\rho+p)\left(u_C^2 + d\right) - \frac{1}{2}K\sigma = 0 \ .
\end{aligned} \tag{44}$$

We treat analogously the equations of the perfect fluid in the bulk. The equation of motion, Eq. (26), can be combined with the continuity equation, Eq. (31), to yield

$$\nabla_\nu\left[\left(1 + \lambda\,{}^{(5)}R\right)(\rho+p)\left(g_{\mu\nu} + u_\mu u_\nu\right)\right] - \left(1 + \lambda\,{}^{(5)}R\right)g_{\mu\nu}\nabla_\nu\rho = 0. \tag{45}$$

Substituting the expression for ${}^{(5)}R$ in Eq. (A19), we integrate both the parallel and the orthogonal components along the N direction to obtain the corresponding boundary conditions on the brane. From the parallel component we find that

$$\begin{aligned}
0 = \int_{-\delta}^{+\delta} dN\nabla_N\Bigg\{&-2\lambda\nabla_A\left[K(\rho+p)\left(g_{AB} + u_Au_B\right)\right] \\
&+\lambda K^2(\rho+p)u_Au_N\left[g_{AB}\left(1 + \frac{1}{d}\right) + \frac{\lambda(\rho+p)}{d\left(M^3_{(5)} - \lambda\rho\right)}\left(du_Au_B - g_{AB}u_C^2\right)\right] \\
&+\left[1 + \lambda\left(R - K^2 - K_{CD}K^{CD} - 2K_{,N}\right)\right](\rho+p)u_Nu_B + 2\lambda Kg_{AB}\nabla_A\rho\Bigg\} \ .
\end{aligned} \tag{46}$$

Here we encounter a third derivative along the N direction of the induced metric g_{AB}. For a continuous metric, the first derivate can be discontinuous, the second derivative can be a delta-like singularity and consequently the third derivative can be a double-peaked delta. When we integrate in N we are left with a term in $K_{,N}$, which is proportional to the second

derivative of the metric and thus potentially singular, evaluated at the end points along the normal direction which define the thickness of the brane. Due to the $\mathbb{Z}_2$–symmetry, however, whereas the delta singularity is even about the brane, with $K(N=-\delta)=-K(N=+\delta)$ and thus $\int dN\nabla_N K = 2K$, the double-peaked delta is odd, with $K_{,N}(N=-\delta)=+K_{,N}(N=+\delta)$ and thus $\int dN\nabla_N\nabla_N K=0$. Consequently, when integrated along N, only odd order derivatives of the metric along N jump across the brane and thereby relating with the singular matter distribution at the location of the brane, while even order derivatives cancel at the end points. However, the term in question contains also the factor $u_A u_N$ which is odd about the brane, thus causing the integral to survive and yield $\int dN\nabla_N\left(K_{,N}u_A u_N\right)=2K_{,N}u_A u_N$ at $N=+\delta$. On the other hand, since the boundary condition in Eq. (42) imposes that either $u_A=0$ or $\rho+p=0$ or $K=0$, then regardless the case this term vanishes on the brane. Then, the parallel component of the boundary condition becomes

$$\begin{aligned}
&- 2\lambda\nabla_A\left[K(\rho+p)\left(g_{AB}+u_A u_B\right)\right]+2\lambda K g_{AB}\nabla_A\rho \\
&+ (\rho+p)u_B u_N\left[1+\lambda\left\{R+\frac{1}{d}K^2\left(1+\frac{\lambda(\rho+p)}{M^3_{P(5)}-\lambda\rho}u_C^2(d-1)\right)-K_{CD}K^{CD}\right\}\right]=0\,. \quad (47)
\end{aligned}$$

Substituting the boundary condition in Eq. (47) back in the parallel projection of Eq. (45), and using also the energy conservation condition in Eq. (41), we find for the induced equation for the fluid on the brane

$$\begin{aligned}
&\nabla_A\left[\left(1+\lambda\left\{R-2K^2-K_{CD}K^{CD}+2K\nabla_N\right\}\right)(\rho+p)\left(g_{AB}+u_A u_B\right)\right] \\
&+ 2\lambda K(\rho+p)\left[u_A u_B\left(K_{AB|C}-K_{AC|B}\right)+u_A u_N\left(K_{CD}K_{DC}g_{AB}+K_{AD}K_{DB}\right)\right] \\
&- \lambda K^2\nabla_A\left[(\rho+p)\left(u_A u_B+g_{AB}\right)\right] \\
&- \lambda K^2\left[g_{AB}\left(1+\frac{1}{d}\right)+\frac{1}{d}\frac{\lambda(\rho+p)}{M^3_{P(5)}-\lambda\rho}\left(du_A u_B-g_{AB}u_C^2\right)\right]\nabla_N\left[(\rho+p)u_A u_N\right] \\
&- \left[2\lambda K\left(K_{AB}-g_{AB}K\right)+g_{AB}\left(1+\lambda\left\{R-K^2-K_{CD}K^{CD}\right\}\right)\right]\nabla_A\rho=0\,. \qquad (48)
\end{aligned}$$

The orthogonal component yields a trivial matching condition because of the continuity conditions across the brane of the quantities involved. We conclude that this component will only be important for the propagation of the fluid off the brane and across the bulk. The propagation on the brane is solely described by Eq. (48), which contains already the continuity condition in Eq. (31), with the possibility of propagation off into the bulk being constrained by the conservation condition in Eq. (41).

These are the induced equations on the brane and can be solved the following way. The effective equations of motion in Eqs. (39) and (48) consist of a coupled system which must

be solved together for the induced metric and for $\nabla_N\left[(\rho+p)\left(u_A u_B+g_{AB}\right)\right]$. With these results, we can then solve the stress-energy conservation condition in Eq. (41), derived from the (AN) component of the Einstein equation, and the (NN) component of the Einstein equation in Eq. (44) together for ρ and the extrinsic curvature, constrained by the matching conditions in Eqs. (36) and (47) which then allow to find the functional form of p in terms of ρ.

From these equations we observe that the coupling of the curvature to the matter Lagrangian density yields a contribution to the effective Newton's constant on the brane. Moreover, it is only through the non-minimal coupling that matter in the bulk interacts with that localized on the brane, in this case the tension σ only. Notice that if $\lambda=0$, i.e. in the absence of a non-minimal coupling of the curvature to the matter Lagrangian density, Eqs. (47) and (48) read respectively

$$\rho+p=0 \tag{49}$$

and

$$\nabla_A(\rho+p)\left(g_{AB}+u_A u_B\right)-g_{AB}\nabla_A\rho=0\ . \tag{50}$$

This means that the presence of a perfect fluid in the bulk space induces on the brane an equation of state characteristic of a cosmological constant, without, however, the quantities ρ and p characterizing the fluid being necessarily constant.

IV. RESULTS

In this section we proceed to study the set of equations derived above for the three particular cases encompassed by the boundary condition derived from the (NN) component of the Einstein equation, Eq. (42).

A. Reflecting Boundary Condition: $u_A=0, \nabla_N u_N=0$

One of the cases that we can consider is that of the brane consisting of a reflecting surface for the incoming fluid flux from the bulk. This translates into setting Dirichlet boundary conditions to the parallel component of the fluid velocity $u_A=0$ and Neumann boundary conditions to the normal component $\nabla_N u_N=0$ at the position of the brane. Moreover,

the $\mathbb{Z}_2$–symmetry implies that $\nabla_N u_A = 0$, whereas boost and translation invariance on the brane implies that $\nabla_B u_A = 0$.

The equations derived in the previous section become as follows. The (AB) and the (NN) components of the Einstein equation are given respectively by

$$\begin{aligned}
& \left(M^3_{P(5)} - \lambda\rho\right)\left[G_{AB} - KK_{AB} + \frac{1}{2}g_{AB}\left(K_{CD}K_{CD} + K^2\right)\right] + g_{AB}\Lambda_{(5)} \\
- & \left[\frac{1}{2}\left(1 + \lambda\left\{R - K^2 - K_{CD}K_{CD}\right\}\right) + \lambda K\nabla_N\right](\rho + p)g_{AB} \\
+ & \left[\frac{1}{2}g_{AB} + \lambda\left(\nabla_A\nabla_B - g_{AB}\nabla^2_C\right)\right]\rho = 0 \ ,
\end{aligned} \tag{51}$$

$$\begin{aligned}
& \left(M^3_{P(5)} - \lambda\rho\right)\frac{1}{2}\left[-R + K^2\left(\frac{2}{d} - 1\right) - K_{CD}K_{CD}\right] + \Lambda_{(5)} \\
+ & \left[\frac{1}{2} - \lambda\nabla^2_C\right]\rho - \lambda K^2(\rho + p) - \frac{1}{2}K\sigma = 0 \ ;
\end{aligned} \tag{52}$$

the parallel component of the induced fluid equation and the condition of stress-energy conservation on the brane respectively by

$$\begin{aligned}
& \nabla_A\left[\left(1 + \lambda\left\{R - 2K^2 - K_{CD}K^{CD} + 2K\nabla_N\right\}\right)(\rho + p)g_{AB}\right] - \lambda K^2 g_{AB}\nabla_A(\rho + p) \\
- & \left[2\lambda K\left(K_{AB} - g_{AB}K\right) + g_{AB}\left(1 + \lambda\left\{R - K^2 - K_{CD}K^{CD}\right\}\right)\right]\nabla_A\rho = 0 \ ,
\end{aligned} \tag{53}$$

$$\frac{1}{d}(d-1)\left(M^3_{P(5)} - \lambda\rho\right)\nabla_B K - \lambda\left(K_{AB} - \frac{1}{d}g_{AB}K\right)\nabla_A\rho = 0 \ . \tag{54}$$

We present also the matching conditions from both the (AB) component of the Einstein equation, Eq. (36), and the parallel component of the fluid equation, Eq. (47), respectively

$$\left(M^3_{P(5)} - \lambda\rho\right)(-K_{AB} + g_{AB}K) + \lambda K(\rho + p)g_{AB} - \lambda g_{AB}\nabla_N\rho = -\frac{1}{2}\sigma g_{AB} \ , \tag{55}$$

and

$$(\rho + p)\nabla_A K + K\nabla_A p = 0 \ . \tag{56}$$

In this case, the extrinsic curvature is intertwined with both the energy density and the pressure of the fluid via the non-minimal coupling λ, being in addition sourced by the brane tension σ. As in the case described next, the role of the brane tension seems superfluous for generating the discontinuity of the bulk geometry at the position of the brane and thus accounting for the singular presence of the brane in the bulk space. Although the system is intricately entangled, we have just enough equations and constraints to be able to solve

it unambiguously given initial conditions. The procedure follows that suggested above for the general system. For further insight into a possible nature of such bulk field, we draw the attention to the next case.

B. Cosmological Constant Fluid: $\rho + p = 0$

Another case contemplated by Eq. (42) is that of the fluid equation of state induced on the brane being $\rho + p = 0$, which corresponds to the bulk fluid inducing a cosmological constant on the brane. The (AB) and (NN) components of the induced Einstein equations become respectively

$$\begin{aligned} & \left(M^3_{P(5)} - \lambda\rho\right)\left[G_{AB} - KK_{AB} + \frac{1}{2}g_{AB}\left(K_{CD}K_{CD} + K^2\right)\right] + g_{AB}\Lambda_{(5)} \\ - & \ \lambda\left(u_Au_B + g_{AB}\right)K\nabla_N(\rho+p) \\ + & \left[\frac{1}{2}g_{AB} + \lambda\left(\nabla_A\nabla_B - g_{AB}\nabla^2_C\right)\right]\rho = 0 \ , \end{aligned} \tag{57}$$

$$\begin{aligned} & \left(M^3_{P(5)} - \lambda\rho\right)\frac{1}{2}\left[-R + K^2\left(\frac{2}{d} - 1\right) - K_{CD}K_{CD}\right] + \Lambda_{(5)} \\ + & \ \lambda u^2_C K\nabla_N(\rho+p) + \left[\frac{1}{2} - \lambda\nabla^2_C\right]\rho - \frac{1}{2}K\sigma = 0 \ , \end{aligned} \tag{58}$$

whereas the parallel component of the fluid equation and the stress-energy conservation condition become respectively

$$\begin{aligned} & \left(1 + \lambda\left\{R - 2K^2 - K_{CD}K^{CD} + 2K\nabla_N\right\}\right)\left[\left(u_Au_B + g_{AB}\right)\nabla_A(\rho+p)\right] \\ - & \ \lambda K^2\left(u_Au_B + g_{AB}\right)\nabla_A(\rho+p) \\ + & \ \lambda\left[2\left(u_Au_B + g_{AB}\right)\nabla_AK + 2K\nabla_A\left(u_Au_B\right) - K^2\left(1 + \frac{1}{d}\right)u_Bu_N\right]\nabla_N(\rho+p) \\ - & \left[2\lambda K\left(K_{AB} - g_{AB}K\right) + g_{AB}\left(1 + \lambda\left\{R - K^2 - K_{CD}K^{CD}\right\}\right)\right]\nabla_A\rho = 0 \ , \end{aligned} \tag{59}$$

$$\begin{aligned} & \frac{1}{d}(d-1)\left(M^3_{P(5)} - \lambda\rho\right)\nabla_BK - \lambda\left(K_{AB} - \frac{1}{d}g_{AB}K\right)\nabla_A\rho \\ - & \ \lambda K\left(u_Au_B - u^2_Cg_{AB}\right)\nabla_A(\rho+p) = 0 \ . \end{aligned} \tag{60}$$

The corresponding matching conditions are

$$\left(M^3_{P(5)} - \lambda\rho\right)\left(-K_{AB} + g_{AB}K\right) - \lambda g_{AB}\nabla_N\rho = -\frac{1}{2}\sigma g_{AB} \tag{61}$$

and

$$\nabla_Ap + u_Au_B\nabla_B(\rho+p) = 0 \ . \tag{62}$$

We now proceed to investigate this case in more detail. Since $\rho + p = 0$ everywhere on the brane, then from boost and translation invariance we must have that $\nabla_A(\rho + p) = 0$ everywhere on the brane. Since both $\rho+p$ and its first derivative along the parallel directions to the brane vanish on the brane, then so must vanish its second derivative. However, terms in $\nabla_N(\rho + p)$ or $\nabla_N\nabla_A(\rho + p)$ do not necessarily vanish. From Eq. (62) it follows that $\nabla_A p = \nabla_A \rho = 0$, and from Eq. (60) that $\nabla_B K = 0$. Moreover, since $\nabla^2(\rho + p) = 0$ implies that $\nabla^2\rho = -\nabla^2 p$, whereas $\nabla\rho = \nabla p$ implies that $\nabla^2\rho = \nabla^2 p$, we must have that $\nabla^2\rho = \nabla^2 p = 0$. We can then solve the Einstein, Eq. (57), and the fluid equation, Eq. (59), iteratively for the evolution of the induced metric and $\nabla_N(\rho+p)$ on the brane, with the sole ambiguity residing in the fluid velocity in the bulk. The value of the extrinsic curvature will then be given by Eq. (58). We also note that with the bulk field behaving on the brane as an effective cosmological constant the role of the brane tension becomes superfluous.

Furthermore, should we allow for $\nabla_A(\rho + p) \neq 0$, then the discontinuity of the extrinsic curvature across the brane would be further sourced by the evolution of $\rho + p$ on the brane and thus generated dynamically according to the equation of motion for the fluid induced on the brane. This would be an interesting idea to pursue further. Despite the increased complexity of the problem, the coupled system would still be solvable.

C. Vanishing Extrinsic Curvature: $K = 0$

The remaining case is that of a vanishing extrinsic curvature, where the (AB) and (NN) components of the effective Einstein equations become

$$\begin{aligned} &\left(M^3_{P(5)} - \lambda\rho\right) G_{AB} - \frac{1}{2}\left(1 + \lambda R\right)\left(u_A u_B + g_{AB}\right)\left(\rho + p\right) + g_{AB}\Lambda_{(5)} \\ +\ &\left[\frac{1}{2} g_{AB} + \lambda\left(\nabla_A\nabla_B - g_{AB}\nabla^2_C\right)\right]\rho = 0\ , \end{aligned} \tag{63}$$

$$-\left(M^3_{P(5)} - \lambda\rho\right)\frac{1}{2}R + \frac{1}{2}\left(1 + \lambda R\right)\left(\rho + p\right)u^2_C + \Lambda_{(5)} + \left[\frac{1}{2} - \lambda\nabla^2_C\right]\rho = 0\ , \tag{64}$$

and the effective equations for the fluid

$$\nabla_A\left[\left(\rho + p\right)\left(u_A u_B + g_{AB}\right)\right] - g_{AB}\nabla_A\rho = 0\ , \tag{65}$$

$$\left(\rho + p\right)\left(u_A u_B + g_{AB}\right)\nabla_A K = 0\ , \tag{66}$$

where we used the Israel matching condition

$$\lambda \nabla_N \rho = \frac{1}{2}\sigma \ . \tag{67}$$

For $K = 0$, the brane tension is supported by the discontinuity in the energy density of the bulk fluid across the two sides about the brane. The matching condition for the fluid equation then yields

$$(\rho + p) u_B u_N \left(1 + \lambda R\right) = 0 \tag{68}$$

and consequently $R = -1/\lambda$. [1] Then Eq. (64) reduces to

$$\nabla^2_C \rho - \frac{1}{\lambda}\left(\frac{M^3_{P(5)}}{\lambda} + \Lambda_{(5)}\right) = 0 \ . \tag{69}$$

This equation can be solved for ρ given initial conditions at the intersection of the brane with the bulk past infinity, obtaining the evolution on the brane of the energy density of the bulk perfect fluid in terms of the parameters of the bulk space. The solution must also be subject to the reproduction of a consistent bulk cosmological constant, in the case of anti-de Sitter $\Lambda_{(5)} < 0$. Upon substitution of Eq. (64), Eq. (63) becomes

$$\left(M^3_{P(5)} - \lambda\rho\right)\left(G_{AB} - \frac{1}{2\lambda} g_{AB}\right) + \lambda \nabla_A \nabla_B \rho = 0 \ . \tag{70}$$

From Eq. (66), we must have that $\nabla_A K = 0$. We can then use the solution for ρ to solve Eq. (70) for g_{AB} and Eq. (65) for p given u_A. Hence, for $K = 0$ the system decouples and can be solved straightforwardly. This case is, however, too limited since it does not constrain the evolution of $\nabla_N \rho$ capable of generating the brane tension σ in Eq. (67).

V. CONCLUSIONS

In this paper we have considered a modified gravity model where the curvature scalar couples non-minimally to the matter Lagrangian density, which here we realize for the case of a perfect fluid. As discussed in Ref. [4], in four dimensions this model can potentially account for the observed rotation curves of galaxies without recourse to dark matter and

[1] The other possibility, $\rho + p = 0$, would be but a particular case of the bulk fluid behaving as a cosmological constant on the brane, as discussed previously.

suggests a solution to the Pioneer anomaly. In addition to this coupling, we have also considered an extra spatial dimension, so that our spacetime is embedded in a five-dimensional bulk space where gravity is described by a five-dimensional Einstein equation.

We find that the resulting model is well defined for the considered physical variables and can be solved for given initial conditions. The new terms that arise from the bulk-brane decomposition yield quite interesting consequences. We found three particular cases which conform to the matching conditions upon the assumption of $\mathbb{Z}_2$–symmetry about the position of the brane and investigated their contribution to the intricately entangled system of equations of motion therein induced. These cases are the following: a) reflecting boundary conditions on the brane, $u_A = \nabla_N u_N = 0$; b) an induced fluid equation of state characteristic of a cosmological constant, $\rho + p = 0$; and c) a vanishing extrinsic curvature, $K = 0$.

Both cases a) and b) seem to render the presence of a brane tension σ superfluous for generating the discontinuity of the bulk geometry at the position of the brane, where the energy density ρ and the pressure p of the fluid can source the discontinuity of the extrinsic curvature in a dynamical manner.

Furthermore, case b) can be regarded as a generalization of a cosmological constant scenario, where the bulk fluid induces on the brane a cosmological constant capable of supporting the presence of the brane. This implies that the evolution on the brane of the energy density of the bulk perfect fluid determines the behaviour of the cosmological constant term on the brane. It is well known that an evolution in terms of cosmic time as $\rho \propto t^{-2}$ is consistent with the value of the vacuum energy density at present [16, 17]. Furthermore, this positive contribution for the brane cosmological constant can have implications for inflation and for the late time acceleration of the universe. This contribution can also have a bearing on the cosmological constant problem since the natural background for fundamental theories, such as supergravity and superstring/M-theory, is the anti-de Sitter space, which on the brane requires a compensating de Sitter contribution. The feasibility of a scenario along these lines will be considered elsewhere.

Finally, case c) allows for the decoupling of the system of equations, with the energy density relating with the bulk cosmological constant $\Lambda_{(5)}$. The presence of the brane is supported by the interaction of the brane tension with the discontinuity across the brane of the fluid energy density, which on the brane is governed dynamically by a completely defined equation for ρ.

APPENDIX A: TENSOR DECOMPOSITION IN GAUSSIAN NORMAL COORDINATES

Here we derive the projection of the Einstein tensor along the parallel and normal directions to the brane, namely G_{AB}, G_{AN} and G_{NN}, using the Gaussian normal prescription [11]

$$
\begin{aligned}
{}^{(5)}\nabla_{\vec{e}_A}\vec{e}_B &= \Gamma^C{}_{BA}\vec{e}_C - K_{AB}\vec{e}_N \ , && \text{(A1)}\\
{}^{(5)}\nabla_{\vec{e}_N}\vec{e}_A &= K_A^C\vec{e}_C \ , && \text{(A2)}\\
{}^{(5)}\nabla_{\vec{e}_A}\vec{e}_N &= K_A^C\vec{e}_C \ . && \text{(A3)}
\end{aligned}
$$

The Gaussian coordinates split the bulk space into a four-dimensional spacetime times a one-dimensional space, so that

$$
{}^{(5)}\mathbf{g} = \left(\begin{array}{c|c} {}^{(4)}\mathbf{g} & \\ \hline & 1 \end{array}\right) \ . \qquad \text{(A4)}
$$

Here, A, $B, \ldots$ denote directions parallel to the brane, while N denotes the orthogonal direction.

We aim to obtain decompositions of the form

$$
{}^{(5)}G_{AB} = \text{terms without extrinsic curvature} + \text{terms with extrinsic curvature} \ , \qquad \text{(A5)}
$$

so that the term without extrinsic curvature will be identified with ${}^{(4)}G_{AB}$. We use the Einstein equations in the bulk to eliminate ${}^{(5)}G_{AB}$ and thus obtain the Einstein equation parallel to the brane. However, this equation will still contain terms in the extrinsic curvature, which can be related to other quantities on the brane through the Israel matching conditions. In this calculation we use geodesic coordinates on the brane, i.e., on the metric ${}^{(4)}g_{AB}$. However, this does not mean that the connection components $\Gamma^C{}_{AB}$ can be ignored, since their derivatives are non-vanishing.

The Riemann tensor can be written as

$$
R(Y,W)Z = \nabla_Y\nabla_W Z - \nabla_W\nabla_Y Z - \nabla_{[Y,W]}Z \ , \qquad \text{(A6)}
$$

where Y, W and Z are vector fields, so that in components:

$$
R^\mu{}_{\alpha\gamma\beta}\vec{e}_\mu = \left(\nabla_{\vec{e}_\gamma}\nabla_{\vec{e}_\beta} - \nabla_{\vec{e}_\beta}\nabla_{\vec{e}_\gamma}\right)\vec{e}_\alpha \ . \qquad \text{(A7)}
$$

On its turn, the Ricci tensor is given by

$$R_{\alpha\beta} = dx^{\gamma}(R^{\mu}{}_{\alpha\gamma\beta}\vec{e}_{\mu}) \ . \tag{A8}$$

Let us now compute the decomposition of R_{AB}. First, one computes ${}^{(5)}R^{J}{}_{ACB}\vec{e}_J$ and ${}^{(5)}R^{J}{}_{ANB}\vec{e}_J$. Noticing that the indices J and L refer to all five dimensions, it follows that

$$\begin{aligned}
{}^{(5)}R^{J}{}_{ACB}\vec{e}_J &= {}^{(5)}\nabla_{\vec{e}_C}\,{}^{(5)}\nabla_{\vec{e}_B}\vec{e}_A - (C \leftrightarrow B) \\
&= {}^{(5)}\nabla_{\vec{e}_C}\left(\Gamma^{D}_{AB}\vec{e}_D - K_{AB}\vec{e}_N\right) - (C \leftrightarrow B) \\
&\doteq \Gamma^{D}_{AB,C}\vec{e}_D - K_{AB,C}\vec{e}_N - K_{AB}K^{D}_{C}\vec{e}_D - (C \leftrightarrow B) \ .
\end{aligned} \tag{A9}$$

The symbol $\doteq$ denotes equality only for the case of geodesic coordinates on the brane. We also note that

$${}^{(4)}\nabla_{\vec{e}_C}(K_{AB}dx^{A} \otimes dx^{B}) \doteq \partial_C\left(K_{AB}\right)dx^{A} \otimes dx^{B}, \tag{A10}$$

which is equivalent to $K_{AB|C} \doteq K_{AB,C}$, where $|C$ denotes ${}^{(4)}\nabla_{\vec{e}_C}$. Then,

$$\begin{aligned}
{}^{(5)}R^{J}{}_{ACB}\vec{e}_J &\doteq \left(\Gamma^{D}_{AB,C} - \Gamma^{D}_{AC,B}\right)\vec{e}_D + \left(K_{AC}K^{D}_{B} - K_{AB}K^{D}_{C}\right)\vec{e}_D \\
&+ \left(K_{AC|B} - K_{AB|C}\right)\vec{e}_N \ .
\end{aligned} \tag{A11}$$

Computing now ${}^{(5)}R^{J}{}_{ANB}\vec{e}_J$ we find that

$$\begin{aligned}
{}^{(5)}R^{J}{}_{ANB}\vec{e}_J &= {}^{(5)}\nabla_{\vec{e}_N}\left(\Gamma^{D}_{AB}\vec{e}_D - K_{AB}\vec{e}_N\right) - {}^{(5)}\nabla_{\vec{e}_B}\,{}^{(5)}\nabla_{\vec{e}_N}\vec{e}_A \\
&\doteq \Gamma^{D}_{AB,N}\vec{e}_D - K_{AB,N}\vec{e}_N - K^{D}_{A|B}\vec{e}_D + K^{D}_{A}K_{DB}\vec{e}_N \ .
\end{aligned} \tag{A12}$$

The five-dimensional Ricci tensor components can now be written as

$$\begin{aligned}
{}^{(5)}R_{AB} &= dx^{L}\left({}^{(5)}R^{J}{}_{ALB}\vec{e}_J\right) = dx^{C}\left({}^{(5)}R^{J}{}_{ACB}\vec{e}_J\right) + dx^{N}\left({}^{(5)}R^{J}{}_{ANB}\vec{e}_J\right) \\
&\doteq \Gamma^{C}_{AB,C} - \Gamma^{C}_{AC,B} + K_{AC}K^{C}_{B} - KK_{AB} - K_{AB,N} + K^{C}_{A}K_{CB} \\
&= {}^{(4)}R_{AB} - KK_{AB} + 2K_{AC}K^{C}_{B} - K_{AB,N} \ ,
\end{aligned} \tag{A13}$$

where $K = K_{AB}\,{}^{(4)}g^{AB}$. Moreover, we compute

$$\begin{aligned}
{}^{(5)}\nabla_{\vec{e}_N}\left(K_{AB}dx^{A}dx^{B}\right) &= K_{AB,N}dx^{A}dx^{B} + K_{AB}dx^{A}\,{}^{(5)}\nabla_{\vec{e}_N}dx^{B} + K_{AB}dx^{B}\,{}^{(5)}\nabla_{\vec{e}_N}dx^{A} \\
&= K_{AB,N}dx^{A}dx^{B} - 2K_{AC}K^{C}_{B}dx^{A}dx^{B} \ ,
\end{aligned} \tag{A14}$$

which yields

$$K_{AB;N} = K_{AB,N} - 2K_{AC}K^{C}_{B} \ . \tag{A15}$$

Finally, ${}^{(5)}R_{AB}$ can be rewritten as

$$ {}^{(5)}R_{AB} = {}^{(4)}R_{AB} - KK_{AB} - K_{AB;N} \ . \tag{A16} $$

Analogously, the other components of the Ricci tensor are given by

$$ \begin{aligned} {}^{(5)}R_{NN} &= -K_{CD}K^{CD} - K_{,N} \ , &\text{(A17)} \\ {}^{(5)}R_{NA} &= K^{B}_{A|B} - K_{|A} \ , &\text{(A18)} \end{aligned} $$

so that the curvature scalar is expressed as

$$ {}^{(5)}R = g^{AB}\,{}^{(5)}R_{AB} + {}^{(5)}R_{NN} = {}^{(4)}R - K^2 - K_{CD}K^{CD} - 2K_{,N} \ . \tag{A19} $$

Then, the decomposition of the five-dimensional Einstein tensor results as follows

$$ \begin{aligned} {}^{(5)}G_{AB} &= {}^{(5)}R_{AB} - \frac{1}{2}\,{}^{(5)}g_{AB}\,{}^{(5)}R \\ &= {}^{(4)}G_{AB} + 2K_{AC}K^{C}_{B} - KK_{AB} - K_{AB,N} \\ &\quad + \frac{1}{2}\,{}^{(4)}g_{AB}\left(K_{CD}K^{CD} + K^2 + 2K_{,N}\right) \ , &\text{(A20)} \\ {}^{(5)}G_{NA} &= {}^{(5)}R_{NA} - \frac{1}{2}\,{}^{(5)}g_{NA}\,{}^{(5)}R \\ &= K^{B}_{A|B} - K_{|A} \ , &\text{(A21)} \\ {}^{(5)}G_{NN} &= {}^{(5)}R_{NN} - \frac{1}{2}\,{}^{(5)}g_{NN}\,{}^{(5)}R \\ &= -\frac{1}{2}\,{}^{(4)}R + \frac{1}{2}\left(K^2 - K_{CD}K^{CD}\right) \ , &\text{(A22)} \end{aligned} $$

Equations (A20 – A22) are known as the Gauss-Codacci relations.

In order to derive the matching conditions in our model, we point out that the Einstein equation, Eq. (7), contains second derivatives of the energy density ρ. We need to express them in Gaussian coordinates. Under a transformation of coordinates $\vec{e}_J = \Lambda^{\nu}_{J}\vec{e}_{\nu}$, where Λ^{ν}_{J} are the components of the versor $\vec{e}_J$ expressed in the $\vec{e}_{\nu}$ basis,

$$ \begin{aligned} \nabla_{\mu}\nabla_{\nu}\rho \quad \longrightarrow \quad & \Lambda^{\mu}_{I}\Lambda^{\nu}_{J}\nabla_{\mu}\nabla_{\nu}\rho \\ &= \Lambda^{\mu}_{I}\nabla_{\mu}\left(\Lambda^{\nu}_{J}\nabla_{\nu}\rho\right) - \Lambda^{\mu}_{I}\left(\nabla_{\mu}\Lambda^{\nu}_{J}\right)\nabla_{\nu}\rho \\ &= \nabla_{I}\nabla_{J}\rho - \Gamma^{L}{}_{JI}\nabla_{L}\rho \ . &\text{(A23)} \end{aligned} $$

Then, terms of the form $\nabla_{\mu}\nabla_{\nu}\rho$ decompose along the directions parallel and orthogonal to the brane as follows

$$ \begin{aligned} &(AB) \qquad \nabla_{A}\nabla_{B}\rho + K_{AB}\nabla_{N}\rho \ , &\text{(A24)} \\ &(AN) \qquad \nabla_{A}\nabla_{N}\rho - K^{B}_{A}\nabla_{B}\rho \ , &\text{(A25)} \\ &(NN) \qquad \nabla_{N}\nabla_{N}\rho \ . &\text{(A26)} \end{aligned} $$

Consequently,

$$^{(5)}\Box\rho = \Box\rho + \nabla_N\nabla_N\rho + K\nabla_N\rho\ . \tag{A27}$$

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